MATB44 Week 2 Notes

1. Exact Equations:

- Suppose we have the following eqn
$M(x,y) + N(x,y)y' = 0.$

If there's a function $F(x,y)$ s.t.
a) $F_x = M(x,y)$ and
b) $F_y = N(x,y)$

then, we call these equations **exact**.

Recall that $dF = F_x\,dx + F_y\,dy$.

$$F_x + F_y \frac{dy}{dx} = 0$$

$$\underbrace{F_x\,dx + F_y\,dy}_{dF} = 0$$

$dF = 0$

$dF = 0$ implies that $F(x,y) = C$ for some constant C.

- **Note:** Finding $F(x,y)$ is the central task in determining if a differential equation is exact and in finding its solution. Since finding $F(x,y)$ is a lengthy process, we can use the below test to see if the differential eqn is exact or not before doing any of the calculation steps.

The test is: If $F(x,y)$ is an exact differential eqn, then $M_y = N_x$.

Proof:

Assume that $F(x,y)$ is an exact differential eqn.

Then:

a) $M = F_x$

b) $N = F_y$

c) $F_{xy} = F_{yx}$ (Mixed derivative rule)

$$
\begin{aligned}
F_{xy} &= (F_x)_y \\
&= M_y
\end{aligned}
$$

$$
\begin{aligned}
F_{yx} &= (F_y)_x \\
&= N_x
\end{aligned}
$$

Therefore, if $F(x,y)$ is an exact differential eqn, then $M_y = N_x$.

- E.g.1 Solve $(y\cos x + 2xe^y) + (\sin x + x^2 e^y - 1)y' = 0$.

Soln:

1. See if $M_y = N_x$.

M_y
$= \partial_y (y\cos x + 2xe^y)$
$= \cos x + 2xe^y$

N_x
$= \partial_x (\sin x + x^2 e^y - 1)$
$= \cos x + 2xe^y$

$M_y = N_x$
This is an exact differential eqn.

2. Let $M = \partial_x F$ and $N = \partial_y F$.
Now,
$\partial_x F = y\cos x + 2xe^y$
$\partial_y F = \sin x + x^2 e^y - 1$

Take the integral of the first eqn to find F.

$$F = \int y\cos x + 2xe^y\,dx$$
$$= \int y\cos x\,dx + \int 2xe^y\,dx$$
$$= y\int \cos x\,dx + e^y\int 2x\,dx$$
$$= y\sin x + x^2e^y + C(y)$$

Note: When taking the integral w.r.t x, you treat y as a constant and vice versa.

Note: C might be a function of y. This is because, when we differentiate w.r.t x, y is treated as a constant.

Now, let's use the second eqn.
$\partial_y F = \sin x + x^2e^y - 1$
Plug the eqn we just got for F into $\partial_y F$.

$$\partial_y(y\sin x + x^2e^y + C(y))$$
$$= \sin x + x^2e^y + C'(y)$$

$$\cancel{\sin x} + \cancel{x^2e^y} + C'(y) = \cancel{\sin x} + \cancel{x^2e^y} - 1$$
$$C'(y) = -1$$

Note: At this point, if you have not made any mistakes, all of the terms with x's should be gone.

$C = \int -1 \, dy$

$= -y + c_1$

$F = y\sin x + x^2 e^y - y + c_1$

Here, any c_1 works. Since it is an additive constant, we can take $c_1 = 0$. If it was multiplicative, we take $c_1 = 1$.

The final answer is

$F = y\sin x + x^2 e^y - y = C$

- E.g. 2 Solve $\left(\frac{y}{x} + 6x\right) + (\log x - 2)y' = 0$

Soln:

$\left(\frac{y}{x} + 6x\right) dx + \left(\log x - 2\right) dy = 0$

$M = \frac{y}{x} + 6x$, $N = \log x - 2$

1. Check to see if $M_y = N_x$

$M_y = \partial_y M$
$= \partial_y \left(y/x + 6x\right)$
$= \frac{1}{x}$

$N_x = \partial_x N$
$= \partial_x (\log x - 2)$
$= \frac{1}{x}$

$M_y = N_x$

F is an exact eqn.

2. Let $M = \partial_x F$ and $N = \partial_y F$ and solve for F.

$$\partial_x F = M$$
$$\begin{aligned} F &= \int M \, dx \\ &= \int y/x + 6x \, dx \\ &= \int y/x \, dx + \int 6x \, dx \\ &= y \int \tfrac{1}{x} \, dx + 6 \int x \, dx \\ &= y \ln|x| + 3x^2 + C(y) \end{aligned}$$

$$\begin{aligned} \partial_y F &= N \\ &= \log x - 2 \end{aligned}$$

$$\begin{aligned} \partial_y F &= \partial_y (y \ln|x| + 3x^2 + C(y)) \\ &= \ln|x| + C'(y) \end{aligned}$$

$$\cancel{\ln|x|} + C'(y) = \cancel{\log x} - 2$$

Note: In this course, we always use $\log_e$. Hence, in this course, $\ln = \log$.

$$C'(y) = -2$$
$$\begin{aligned} C &= \int -2 \, dy \\ &= -2y + C_1 \end{aligned}$$

$$F = y \ln|x| + 3x^2 - 2y + C_1$$

Let $C_1 = 0$

$$F = y \ln|x| + 3x^2 - 2y = C$$

- E.g. 3 Solve $(2xy - 9x^2) + (2y + x^2 + 1)y' = 0$

Soln:

$(2xy - 9x^2)\,dx + (2y + x^2 + 1)\,dy = 0$

$M = 2xy - 9x^2, \quad N = 2y + x^2 + 1$

1. Check if $M_y = N_x$.

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y (2xy - 9x^2) \\ &= 2x \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (2y + x^2 + 1) \\ &= 2x \end{aligned}$$

$M_y = N_x$

Hence, F is an exact differential eqn.

2. Let $M = \partial_x F$ and $N = \partial_y F$ and Solve for F.

$$\begin{aligned} \partial_x F &= M \\ &= 2xy - 9x^2 \\ F &= \int 2xy - 9x^2 \, dx \\ &= x^2 y - 3x^3 + c(y) \end{aligned}$$

$\partial_y F = N$

$$\begin{aligned} RHS &= N \\ &= 2y + x^2 + 1 \end{aligned}$$

$$\begin{aligned} LHS &= \partial_y F \\ &= \partial_y (x^2 y - 3x^3 + c(y)) \\ &= x^2 + c'(y) \end{aligned}$$

$$\begin{aligned} LHS &= RHS \\ x^2 + c'(y) &= 2y + x^2 + 1 \\ c'(y) &= 2y + 1 \\ C &= \int 2y + 1 \, dy \\ &= y^2 + y + c_1 \end{aligned}$$

$F = x^2 y - 3x^3 + y^2 + y + c_1$

Let $c_1 = 0$

$F = y^2 + (x^2+1)y - 3x^3 = C$

2. Exact Equation with Integrating Factor:

- So far, all the eqns we've seen have $M_y = N_x$. Not all eqns are like this. If $M_y \neq N_x$, we need to multiply both sides of the eqn by μ, the integrating factor.

- E.g. 4 Solve $(3xy + y^2) + (x^2 + xy)y' = 0$

Soln:

$(3xy + y^2) + (x^2 + xy)y' = 0$

$\rightarrow \underbrace{(3xy + y^2)}_{M} dx + \underbrace{(x^2 + xy)}_{N} dy = 0$

1. Check if $M_y = N_x$.

$$\begin{aligned} M_y &= \partial_y M \\ &= \partial_y (3xy + y^2) \\ &= 3x + 2y \end{aligned}$$

$$\begin{aligned} N_x &= \partial_x N \\ &= \partial_x (x^2 + xy) \\ &= 2x + y \end{aligned}$$

Here, $M_y \neq N_x$

2. Multiply both sides of the eqn by $\mu(x)$.

$$\mu(x)(3xy + y^2) \quad dx + \mu(x)(x^2 + xy)\, dy = 0$$

3. Solve for $\mu(x)$.

To solve for $\mu(x)$, we assume that $\partial_y[\mu(x) M] = \partial_x[\mu(x) N]$.

$$\partial_y[\mu(x) M] = \mu(x)[\partial_y M]$$

$$\partial_x[\mu(x) N] = \mu'(x) N + \mu(x)[\partial_x N]$$

$$\mu(x)(\partial_y M) = \mu'(x)N + \mu(x)(\partial_x N)$$

$$\mu'(x)N = \mu(x)(\partial_y M) - \mu(x)(\partial_x N)$$

$$= \mu(x)[\partial_y M - \partial_x N]$$

$$\frac{\mu'(x)}{\mu(x)} = \frac{\partial_y M - \partial_x N}{N}$$

Note: For this to be solvable, $\frac{\partial_y M - \partial_x N}{N}$ CANNOT have any "y" in it.

I.e. $\frac{\partial_y M - \partial_x N}{N}$ depends on "x" only.

$$\partial_y M = 3x + 2y$$

$$\partial_x N = 2x + y$$

$$N = x^2 + xy$$

$$\frac{\partial_y M - \partial_x N}{N}$$

$$= \frac{3x + 2y - (2x+y)}{x^2+xy}$$

$$= \frac{3x+2y-2x-y}{x^2+xy}$$

$$= \frac{x+y}{x(x+y)}$$

$$= \frac{1}{x}$$ ← No "y"'s left.

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{x} \longrightarrow$$ This is separable eqn.

$$\frac{\mu'(x)}{\mu(x)} = \frac{1}{x}$$

$$\frac{1}{\mu(x)}\, d\mu(x) = \frac{1}{x}\, dx$$

$$(\ln(\mu(x)))' = \frac{1}{x}$$

$$\int (\ln(\mu(x)))'\, dx = \int \frac{1}{x}\, dx$$

$$\ln(\mu(x)) + C_1 = \ln|x| + C_2$$

$$\ln(\mu(x)) = \ln|x| + C_2 - C_1$$

$$= \ln|x| + C$$

$$\mu(x) = e^{\ln|x| + C}$$

$$= e^{\ln|x|} \cdot e^{C}$$

Let $c' = e^{c}$

$$\mu(x) = x \cdot c'$$

Let $c' = 1$

$$\mu(x) = x$$

So now we have:

$$x(3xy + y^2)\, dx + x(x^2 + xy)\, dy = 0$$

Note: After we find $\mu(x)$, we don't need to show that $\partial_y[\mu(x) M] = \partial_x[\mu(x) N]$.

4. Solve for F.

Now, we solve for F using the steps from when F is an exact eqn from the start.

$M_1 = x(3xy + y^2)$
$N_1 = x(x^2 + xy)$

Let $M_1 = \partial_x F$ and $N_1 = \partial_y F$.
Now,
$\partial_x F = x(3xy + y^2)$
$\partial_y F = x(x^2 + xy)$

Take the integral of $\partial_x F$ w.r.t. x to find F.

$$F = \int x(3xy + y^2)\,dx$$
$$= \int 3x^2y + xy^2\,dx$$
$$= \int 3x^2y\,dx + \int xy^2\,dx$$
$$= x^3y + \frac{x^2y^2}{2} + C(y)$$

Now, we will use the second eqn.

$$\partial_y F = x^3 + x^2y$$
Also $\partial_y F = \partial_y \left(x^3y + \frac{x^2y^2}{2} + C(y)\right)$
$$= x^3 + x^2y + C'(y)$$
$$x^3 + x^2y + C'(y) = x^3 + x^2y$$
$$C'(y) = 0$$
$$C = \int 0\,dy$$
$$= 0$$

$$F = x^3y + \frac{x^2y^2}{2} = c$$

Note: If the question is "Verify that this eqn has an integrating factor that depends on x only.", you must end up with something without y.

Note: If the question is "Check whether this eqn has an integrating factor that depends on x only.", it's trickier than the previous question. In the previous question, you know right away that the integrating factor depends on x only, where here, you don't.

- E.g. 5. Find $\mu(y)$ and solve
$y + (2xy - e^{-2y})y' = 0$.

Soln:

$$y\,dx + (2xy - e^{-2y})\,dy = 0$$

1. Multiply both sides of the eqn by $\mu(y)$ and solve for $\mu(y)$.

$\mu(y)\,y\,dx + \mu(y)(2xy - e^{-2y})\,dy = 0$

Now, assume that $\partial_y[\mu(y)M] = \partial_x[\mu(y)N]$.

$\partial_y[\mu(y)M]$
$= \mu'(y)M + \mu(y)[\partial_y M]$

$\partial_x[\mu(y)N]$
$= \mu(y)(\partial_x N)$

$$\mu'(y) M + \mu(y)[\partial_y M] = \mu(y)(\partial_x N)$$
$$\mu'(y) M = \mu(y)(\partial_x N) - \mu(y)[\partial_y M]$$
$$= \mu(y)[\partial_x N - \partial_y M]$$
$$\frac{\mu'(y)}{\mu(y)} = \frac{\partial_x N - \partial_y M}{M}$$

Note: In the previous example, μ is a function of x. At this step, I said that there should be no "y"'s after you simplify $\frac{\partial_y M - \partial_x N}{N}$.

However, since μ is a function of y in this example, there should be no "x"'s after you simplify $\frac{\partial_x N - \partial_y M}{M}$.

$$\partial_y M = \partial_y (y)$$
$$= 1$$

$$\partial_x N = \partial_x (2xy - e^{-2y})$$
$$= 2y$$

$$\frac{\partial_x N - \partial_y M}{M} = \frac{2y - 1}{y}$$
$$= 2 - \frac{1}{y}$$

$$\frac{\mu'(y)}{\mu(y)} = 2 - \frac{1}{y}$$

$$(\ln(\mu(y)))' = 2 - \frac{1}{y}$$

$$\int (\ln(\mu(y)))' \, dy = \int 2 - \frac{1}{y} \, dy$$

$$\ln(\mu(y)) + c_1 = 2y - \ln|y| + c_2$$

$$\ln(\mu(y)) = 2y - \ln|y| + c_2 - c_1$$

$$= 2y - \ln|y| + c$$

$$\mu(y) = e^{2y - \ln|y| + c}$$

$$= \frac{e^{2y} \cdot e^{c}}{e^{\ln|y|}}$$

let $c' = e^c$

$$\mu(y) = \frac{e^{2y} \cdot c'}{y}$$

let $c' = 1$

$$\mu(y) = \frac{e^{2y}}{y}$$

2. Solve for F.

The eqn we now have is:

$$\underbrace{\frac{e^{2y}}{y} \cdot y}_{M_1} + \underbrace{\frac{e^{2y}}{y}\left(2xy - e^{-2y}\right)}_{N_1}$$

let $\partial_x F = M_1$

$$\partial_x F = e^{2y}$$

$$F = \int e^{2y} \, dx$$

$$= e^{2y} x + c(y)$$

$\partial_y F = N_1$

$= \frac{e^{2y}}{y}\left(2xy - e^{-2y}\right)$

Also $\partial_y F = \partial_y (e^{2y}x + C(y))$

$= 2xe^{2y} + C'(y)$

$2xe^{2y} + C'(y) = \frac{e^{2y}}{y}\left(2xy - e^{-2y}\right)$

$= 2xe^{2y} - \frac{1}{y}$

$C'(y) = -\frac{1}{y}$

$C = \int \frac{-1}{y}\, dy$

$= -\log|y| + C_1$

Let $C_1 = 0$

$F = e^{2y}x - \log|y| = C$

3. More Examples

a) Solve $(2x+3) + (2y-2)y' = 0$

Soln:

$(2x+3)\,dx + (2y-2)\,dy = 0$

1. Check if $M_y = N_x$

$M = 2x+3$ $N = 2y-2$

[redacted]

$\partial_y M = \partial_y (2x+3)$

$= 0$

$\partial_x N = 0$

$M_y = N_x$

This is an exact equation.

2. Let $\partial_x F = M$ and $\partial_y F = N$.

$\partial_x F = 2x+3$
$F = \int 2x+3\,dx$
$= x^2 + c(y) + 3x$

$\partial_y F = N$
$= 2y-2$
Also $\partial_y F = \partial_y(x^2 + c(y) + 3x)$
$= c'(y)$

$c'(y) = 2y-2$
$c = \int 2y-2\,dy$
$= y^2 - 2y + c_1$
Let $c_1 = 0$

$F = x^2+3x+y^2-2y = c$

b) Solve $(3x^2 - 2xy + 2) + (6y^2 - x^2 + 3)y' = 0$
Soln:

$(3x^2 - 2xy + 2)\,dx + (6y^2 - x^2 + 3)\,dy = 0$
$M = 3x^2 - 2xy + 2$
$N = 6y^2 - x^2 + 3$
1. Check if $M_y = N_x$

$M_y = \partial_y M$
$= \partial_y(3x^2 - 2xy + 2)$
$= -2x$

$N_x = \partial_x N$
$= \partial_x(6y^2 - x^2 + 3)$
$= -2x$

$M_y = N_x$
Hence, this is an exact equation.

2. Let $\partial_x F = M$ and $\partial_y F = N$ and Solve for F.

$$\partial_x F = M$$
$$= 3x^2 - 2xy + 2$$
$$F = \int 3x^2 - 2xy + 2 \, dx$$
$$= x^3 - x^2y + 2x + c(y)$$

$$\partial_y F = N$$
$$= 6y^2 - x^2 + 3$$

Also $\partial_y F = \partial_y (x^3 - x^2y + 2x + c(y))$
$$= -x^2 + c'(y)$$

$$-x^2 + c'(y) = 6y^2 - x^2 + 3$$
$$c'(y) = 6y^2 + 3$$
$$c = \int 6y^2 + 3 \, dy$$
$$= 2y^3 + 3y + c_1$$

Let $c_1 = 0$

$$F = x^3 - x^2y + 2x + 2y^3 + 3y = c$$

c) Solve $(3x^2y + 2xy + y^3) + (x^2+y^2)y' = 0$

Soln:

$(3x^2y + 2xy + y^3)\,dx + (x^2+y^2)\,dy = 0$

$M = 3x^2y + 2xy + y^3$

$N = x^2 + y^2$

1. Check if $M_y = N_x$

$$M_y = \partial_y M$$
$$= \partial_y (3x^2y + 2xy + y^3)$$
$$= 3x^2 + 2x + 3y^2$$

$$N_x = \partial_x N$$
$$= \partial_x (x^2 + y^2)$$
$$= 2x$$

$M_y \neq N_x \leftarrow$ This is not an exact equation.

2. Multiply both sides of the eqn by μ.

$$\mu(3x^2y + 2xy + y^3)\,dx + \mu(x^2+y^2)\,dy = 0$$

3. Assume that $\partial_y(\mu M) = \partial_x(\mu N)$

$$\partial_y(\mu M) = \mu(\partial_y M)$$
$$\partial_x(\mu N) = \mu' N + \mu(\partial_x N)$$

$$\mu(\partial_y M) = \mu' N + \mu(\partial_x N)$$
$$\mu(\partial_y M) - \mu(\partial_x N) = \mu' N$$
$$\mu(\partial_y M - \partial_x N) = \mu' N$$
$$\frac{\mu'}{\mu} = \frac{\partial_y M - \partial_x N}{N}$$

$$\partial_y M = 3x^2 + 2x + 3y^2$$
$$\partial_x N = 2x$$

$$\frac{\mu'}{\mu} = \frac{3x^2 + 2x + 3y^2 - 2x}{x^2+y^2}$$
$$= \frac{3(x^2+y^2)}{x^2+y^2}$$
$$= 3$$

$(\ln(\mu))' = 3$

$\int (\ln(\mu))' \, dx = \int 3 \, dx$

$\ln(\mu) = 3x + c$

$\mu = e^{3x+c}$

$= e^{3x} \cdot e^{c}$

let $c' = e^{c}$.

$\mu = e^{3x} \cdot c'$

Let $c' = 1$

$\mu = e^{3x}$

4. Assume the question is like a normal exact eqn question and solve for F.

$$\underbrace{e^{3x}(3x^2y + 2xy + y^3)}_{M_1}dx + \underbrace{e^{3x}(x^2+y^2)}_{N_1}dy = 0$$

let $\partial_x F = M_1$

let $\partial_y F = N_1$

$\partial_x F = M_1$

$= e^{3x}(3x^2y + 2xy + y^3)$

$F = \int e^{3x}(3x^2y + 2xy + y^3)\, dx$

$= \dfrac{y(3x^2+y^2)e^{3x}}{3} + c(y)$

$$\partial_y F = N_1$$
$$= e^{3x}(x^2+y^2)$$

Also, $\partial_y F = \partial_y \left(\frac{y(3x^2+y^2)e^{3x}}{3} + c(y) \right)$

$$= \partial_y \left(\frac{y(3x^2+y^2)e^{3x}}{3} \right) + \partial_y (c(y))$$

$$= \frac{e^{3x}}{3} \left(\partial_y (3x^2 y + y^3) \right) + c'(y)$$
$$= \frac{e^{3x}}{3} \left(3x^2 + 3y^2 \right) + c'(y)$$
$$= e^{3x}(x^2+y^2) + c'(y)$$

$$e^{3x}(x^2+y^2) + c'(y) = e^{3x}(x^2+y^2)$$
$$c'(y) = 0$$
$$c = 0$$

$$F = \frac{y(3x^2+y^2)e^{3x}}{3} = C$$

d) Find an integrating factor depending on xy and solve the eqn

$$\left(3x + \frac{6}{y}\right) + \left(\frac{x^2}{y} + \frac{3y}{x}\right)y' = 0.$$

Soln:

$$\left(3x + \frac{6}{y}\right) dx + \left(\frac{x^2}{y} + \frac{3y}{x}\right) dy = 0$$

$$M = 3x + \frac{6}{y}$$

$$N = \frac{x^2}{y} + \frac{3y}{x}$$

1. Multiply both sides of the eqn by $\mu(xy)$.

$$\mu(xy)\left(3x + \frac{6}{y}\right) dx + \mu(xy)\left(\frac{x^2}{y} + \frac{3y}{x}\right) dy = 0$$

2. Assume that $\partial_y(\mu M) = \partial_x(\mu N)$.

$$\begin{aligned}\partial_y(\mu M) &= \partial_y(\mu(xy)M) \\ &= x\mu'(xy)M + \mu(xy)(\partial_y M)\end{aligned}$$

$$\begin{aligned}\partial(\mu N) &= \partial_x(\mu(xy)N) \\ &= y\mu'(xy)N + \mu(xy)(\partial_x N)\end{aligned}$$

$$x\mu'(xy)M + \mu(xy)(\partial_y M) = y\mu'(xy)N + \mu(xy)(\partial_x N)$$
$$x\mu'(xy)M - y\mu'(xy)N = \mu(xy)(\partial_x N) - \mu(xy)(\partial_y M)$$
$$\mu'(xy)[xM - yN] = \mu(xy)[\partial_x N - \partial_y M]$$
$$\frac{\mu'(xy)}{\mu(xy)} = \frac{\partial_x N - \partial_y M}{xM - yN}$$

We need $\dfrac{\partial_x N - \partial_y M}{xM - yN}$ to depend on xy only.

$$\partial_x N = \partial_x\left(\frac{x^2}{y} + \frac{3y}{x}\right)$$
$$= \frac{2x}{y} - \frac{3y}{x^2}$$

$$\partial_y M = \partial_y\left(3x + \frac{6}{y}\right)$$
$$= \frac{-6}{y^2}$$

$$\frac{\partial_x N - \partial_y M}{xM - yN} = \frac{1}{xy}$$

$$\frac{\mu'(xy)}{\mu(xy)} = \frac{1}{xy}$$
$$(\ln(\mu(xy)))' = \frac{1}{xy}$$
let $t = xy$
$$(\ln(\mu(t)))' = \frac{1}{t}$$
$$\int (\ln(\mu(t)))'\,dt = \int \frac{1}{t}\,dt$$

$\ln(\mu(t)) + C_1 = \ln|t| + C_2$

$\ln(\mu(t)) = \ln|t| + C_2 - C_1$

$= \ln|t| + c$

$\mu(t) = e^{\ln|t| + c}$

$= e^{\ln|t|} \cdot e^{c}$

Let $c' = e^{c}$

$\mu(t) = t \cdot c'$

Let $c' = 1$

$\mu(t) = t$

$= xy$

3. Treat the eqn as an exact eqn and solve for F.

$\mu(xy) M\,dx + \mu(xy) N\,dy = 0$

$$\underbrace{(xy)\left(3x + \frac{6}{y}\right)}_{M_1} dx + \underbrace{(xy)\left(\frac{x^2}{y} + \frac{3y}{x}\right)}_{N_1} dy = 0$$

Let $\partial_x F = M_1$

Let $\partial_y F = N_1$

$\partial_x F = M_1$

$= (xy)\left(3x + \frac{6}{y}\right)$

$F = \int 3x^2 y + 6x\,dx$

$= x^3 y + 3x^2 + C(y)$

$$\partial_y F = N_1$$
$$= x^3 + 3y^2$$

Also $\partial_y F = \partial_y (x^3 y + 3x^2 + c(y))$
$$= x^3 + c'(y)$$

$$x^3 + c'(y) = x^3 + 3y^2$$
$$c'(y) = 3y^2$$
$$c = \int 3y^2 \, dy$$
$$= y^3 + c_1$$
let $c_1 = 0$

$$F = x^3 y + 3x^2 + y^3 = C$$